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福建师范大学数学与计算机科学学院

# 关于正交矩阵群的有限子群

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2006 年 12 月 2 日

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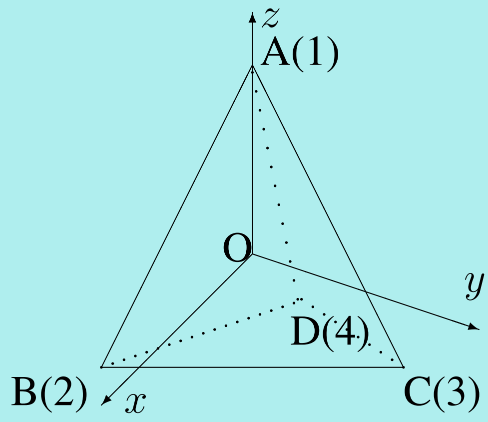
# 一. 正四面体的对称群




例1 在上图正四面体中, 设$\overline{AB}=\overline{AC}=\overline{AD}=\overline{BC}=\overline{BD}=\overline{CD}=1$, 选正四面体的重心$O$为原点, 在$\Delta OAC$, 过$O$作一直线$\overline{OC'}$使得$\overline{OC'}\perp\overline{OA}$, 以$\overrightarrow{OA}$为$z-$轴, $\overrightarrow{OC'}$为$y-$轴, 向量积$\overrightarrow{OC'}\times\overrightarrow{OA}$为$x-$轴. 设$K, I, J$分别为$\overline{AB}$, $\overline{AC}$, $\overline{AD}$的中点. 计算得,

$$\begin{aligned}
&\alpha_1=\overrightarrow{OA}=(0,0,\tfrac{\sqrt{6}}{4})^T, && \alpha_2=\overrightarrow{OB}=(\tfrac{1}{2},-\tfrac{\sqrt{3}}{6},-\tfrac{\sqrt{6}}{12})^T,\\
&\alpha_3=\overrightarrow{OC}=(0,\tfrac{\sqrt{3}}{3},-\tfrac{\sqrt{6}}{12})^T, && \alpha_4=\overrightarrow{OD}=(-\tfrac{1}{2},-\tfrac{\sqrt{3}}{6},-\tfrac{\sqrt{6}}{12})^T,\\
&\alpha_5=\overrightarrow{OI}=(0,\tfrac{1}{6}\sqrt{3},\tfrac{1}{12}\sqrt{6})^T, && \alpha_6=\overrightarrow{OJ}=(-\tfrac{1}{4},-\tfrac{1}{12}\sqrt{3},\tfrac{1}{12}\sqrt{6})^T,\\
&\alpha_7=\overrightarrow{OK}=(\tfrac{1}{4},-\tfrac{1}{12}\sqrt{3},\tfrac{1}{12}\sqrt{6})^T, && \overline{\alpha}_5=-\alpha_5,\\
&\overline{\alpha}_6=-\alpha_6, && \overline{\alpha}_7=-\alpha_7.
\end{aligned}$$

## 正四面体的旋转群

$$H=\{I,A_1,A_1^2,A_2,A_2^2,A_3,A_3^2,A_4,A_4^2,B_1,B_2,B_3\},$$

其中

$$I=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\qquad A_1=\begin{pmatrix}-\frac{1}{2}&-\frac{1}{2}\sqrt{3}&0\\\frac{1}{2}\sqrt{3}&-\frac{1}{2}&0\\0&0&1\end{pmatrix},$$

$$A_1^2=\begin{pmatrix}-\frac{1}{2}&\frac{1}{2}\sqrt{3}&0\\-\frac{1}{2}\sqrt{3}&-\frac{1}{2}&0\\0&0&1\end{pmatrix},\qquad A_2=\begin{pmatrix}\frac{1}{2}&-\frac{1}{6}\sqrt{3}&-\frac{1}{3}\sqrt{6}\\-\frac{1}{2}\sqrt{3}&-\frac{1}{6}&-\frac{1}{3}\sqrt{2}\\0&\frac{2}{3}\sqrt{2}&-\frac{1}{3}\end{pmatrix},$$

$$A_2^2=\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\sqrt{3}&0\\-\frac{1}{6}\sqrt{3}&-\frac{1}{6}&\frac{2}{3}\sqrt{2}\\-\frac{1}{3}\sqrt{6}&-\frac{1}{3}\sqrt{2}&-\frac{1}{3}\end{pmatrix},\qquad A_3=\begin{pmatrix}-\frac{1}{2}&\frac{1}{6}\sqrt{3}&\frac{1}{3}\sqrt{6}\\-\frac{1}{6}\sqrt{3}&\frac{5}{6}&-\frac{1}{3}\sqrt{2}\\-\frac{1}{3}\sqrt{6}&-\frac{1}{3}\sqrt{2}&-\frac{1}{3}\end{pmatrix},$$

$$A_3^2=\begin{pmatrix}-\frac{1}{2}&-\frac{1}{6}\sqrt{3}&-\frac{1}{3}\sqrt{6}\\\frac{1}{6}\sqrt{3}&\frac{5}{6}&-\frac{1}{3}\sqrt{2}\\\frac{1}{3}\sqrt{6}&-\frac{1}{3}\sqrt{2}&-\frac{1}{3}\end{pmatrix},\qquad A_4=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}\sqrt{3}&0\\\frac{1}{6}\sqrt{3}&-\frac{1}{6}&\frac{2}{3}\sqrt{2}\\\frac{1}{3}\sqrt{6}&-\frac{1}{3}\sqrt{2}&-\frac{1}{3}\end{pmatrix},$$

$$A_4^2=\begin{pmatrix}\frac{1}{2}&\frac{1}{6}\sqrt{3}&\frac{1}{3}\sqrt{6}\\\frac{1}{2}\sqrt{3}&-\frac{1}{6}&-\frac{1}{3}\sqrt{2}\\0&\frac{2}{3}\sqrt{2}&-\frac{1}{3}\end{pmatrix},\qquad B_1=\begin{pmatrix}-1&0&0\\0&\frac{1}{3}&\frac{2}{3}\sqrt{2}\\0&\frac{2}{3}\sqrt{2}&-\frac{1}{3}\end{pmatrix},$$

$$B_2=\begin{pmatrix}0&\frac{1}{3}\sqrt{3}&-\frac{1}{3}\sqrt{6}\\\frac{1}{3}\sqrt{3}&-\frac{2}{3}&-\frac{1}{3}\sqrt{2}\\-\frac{1}{3}\sqrt{6}&-\frac{1}{3}\sqrt{2}&-\frac{1}{3}\end{pmatrix},\qquad B_3=\begin{pmatrix}0&-\frac{1}{3}\sqrt{3}&\frac{1}{3}\sqrt{6}\\-\frac{1}{3}\sqrt{3}&-\frac{2}{3}&-\frac{1}{3}\sqrt{2}\\\frac{1}{3}\sqrt{6}&-\frac{1}{3}\sqrt{2}&-\frac{1}{3}\end{pmatrix}.$$

它的乘法表:

| $\cdot$ | $I$ | $A_1$ | $A_1^2$ | $A_2$ | $A_2^2$ | $A_3$ | $A_3^2$ | $A_4$ | $A_4^2$ | $B_1$ | $B_2$ | $B_3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $I$ | $I$ | $A_1$ | $A_1^2$ | $A_2$ | $A_2^2$ | $A_3$ | $A_3^2$ | $A_4$ | $A_4^2$ | $B_1$ | $B_2$ | $B_3$ |
| $A_1$ | $A_1$ | $A_1^2$ | $I$ | $A_4^2$ | $B_2$ | $A_2^2$ | $B_3$ | $A_3^2$ | $B_1$ | $A_2$ | $A_3$ | $A_4$ |
| $A_1^2$ | $A_1^2$ | $I$ | $A_1$ | $B_1$ | $A_3$ | $B_2$ | $A_4$ | $B_3$ | $A_2$ | $A_4^2$ | $A_2^2$ | $A_3^2$ |
| $A_2$ | $A_2$ | $A_3^2$ | $B_2$ | $A_2^2$ | $I$ | $A_4^2$ | $B_1$ | $A_1^2$ | $B_3$ | $A_1$ | $A_4$ | $A_3$ |
| $A_2^2$ | $A_2^2$ | $B_1$ | $A_4$ | $I$ | $A_2$ | $B_3$ | $A_1$ | $B_2$ | $A_3$ | $A_3^2$ | $A_1^2$ | $A_4^2$ |
| $A_3$ | $A_3$ | $A_4^2$ | $B_3$ | $A_1^2$ | $B_1$ | $A_3^2$ | $I$ | $A_2^2$ | $B_2$ | $A_4$ | $A_1$ | $A_2$ |
| $A_3^2$ | $A_3^2$ | $B_2$ | $A_2$ | $B_3$ | $A_4$ | $I$ | $A_3$ | $B_1$ | $A_1$ | $A_2^2$ | $A_4^2$ | $A_1^2$ |
| $A_4$ | $A_4$ | $A_2^2$ | $B_1$ | $A_3^2$ | $B_3$ | $A_1^2$ | $B_2$ | $A_4^2$ | $I$ | $A_3$ | $A_2$ | $A_1$ |
| $A_4^2$ | $A_4^2$ | $B_3$ | $A_3$ | $B_2$ | $A_1$ | $B_1$ | $A_2$ | $I$ | $A_4$ | $A_1^2$ | $A_3^2$ | $A_2^2$ |
| $B_1$ | $B_1$ | $A_4$ | $A_2^2$ | $A_3$ | $A_1^2$ | $A_2$ | $A_4^2$ | $A_1$ | $A_3^2$ | $I$ | $B_3$ | $B_2$ |
| $B_2$ | $B_2$ | $A_2$ | $A_3^2$ | $A_1$ | $A_4^2$ | $A_4$ | $A_1^2$ | $A_3$ | $A_2^2$ | $B_3$ | $I$ | $B_1$ |
| $B_3$ | $B_3$ | $A_3$ | $A_4^2$ | $A_4$ | $A_3^2$ | $A_1$ | $A_2^2$ | $A_2$ | $A_1^2$ | $B_2$ | $B_1$ | $I$ |

4元交错群

$$\mathbb{A}_4 = \left\{ \begin{array}{cccc} (1), & (234), & (243), & (134), \\ (143), & (124), & (142), & (123), \\ (132), & (13)(24), & (14)(23), & (12)(34) \end{array} \right\}$$

同构映射:

$$\begin{array}{rclrcl} \psi : H & \longrightarrow & \mathbb{A}_4, & & & \\ A_1 & \longmapsto & (234), & A_1^2 & \longmapsto & (243), \\ A_2 & \longmapsto & (134), & A_2^2 & \longmapsto & (143), \\ A_3 & \longmapsto & (124), & A_3^2 & \longmapsto & (142), \\ A_4 & \longmapsto & (123), & A_4^2 & \longmapsto & (132), \\ B_1 & \longmapsto & (13)(24), & B_2 & \longmapsto & (14)(23), \\ B_3 & \longmapsto & (12)(34), & I & \longmapsto & (1). \end{array}$$

# 二. 群作用的例子

**定义** 设$G$是一个群, $\Omega$是一个非空集合. 如果有一个映射

$$\circ : G \times \Omega \to \Omega, (a, x) \mapsto a \circ x, \forall a \in G, x \in \Omega$$

满足$(ab) \circ x = a \circ (b \circ x)$, $e \circ x = x$, $\forall a, b \in G$, $\forall x \in \Omega$, 则称群$G$在集合$\Omega$上有一个**作用**.

**例** 设$\Omega = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \overline{\alpha}_5, \overline{\alpha}_6, \overline{\alpha}_7\}$, 则

$$\{M\alpha_i, M\overline{\alpha}_j | M \in H, i = 1, 2, 3, 4, 5, 6, 7, j = 5, 6, 7\} = \Omega,$$

$$\{M\alpha_i, M\overline{\alpha}_j | M \in G, i = 1, 2, 3, 4, 5, 6, 7, j = 5, 6, 7\} = \Omega.$$

定义 设群$G$在集合$\Omega$上有一个作用, 对于$x \in \Omega$, 令$G(x) = \{gx | x \in G\}$, 称$G(x)$是$x$的轨道.

例如, 若$\Omega = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \overline{\alpha}_5, \overline{\alpha}_6, \overline{\alpha}_7\}$, 则

$$H(\alpha_1) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\},$$

$$H(\alpha_5) = \{\alpha_5, \alpha_6, \alpha_7, \overline{\alpha}_5, \overline{\alpha}_6, \overline{\alpha}_7\},$$

且

$$\Omega = H(\alpha_1) \cup H(\alpha_5),\ H(\alpha_1) \cap H(\alpha_5) = \emptyset$$

例如, 令

$$\Omega=\{\beta^n_{1,2},\beta^n_{2,1},\beta^n_{1,3},\beta^n_{3,1},\beta^n_{1,4},\beta^n_{4,1},\beta^n_{2,3},\beta^n_{3,2},\beta^n_{2,4},\beta^n_{4,2},\beta^n_{3,4},\beta^n_{4,3}|n=1,2,\cdots\},$$

其中

$$\begin{array}{ll}
\beta^n_{1,2}=\frac{1}{n}\alpha_1+(1-\frac{1}{n})\alpha_2, & \beta^n_{2,1}=\frac{1}{n}\alpha_2+(1-\frac{1}{n})\alpha_1,\\
\beta^n_{1,3}=\frac{1}{n}\alpha_1+(1-\frac{1}{n})\alpha_3, & \beta^n_{3,1}=\frac{1}{n}\alpha_3+(1-\frac{1}{n})\alpha_1,\\
\beta^n_{1,4}=\frac{1}{n}\alpha_1+(1-\frac{1}{n})\alpha_4, & \beta^n_{4,1}=\frac{1}{n}\alpha_4+(1-\frac{1}{n})\alpha_1,\\
\beta^n_{2,3}=\frac{1}{n}\alpha_2+(1-\frac{1}{n})\alpha_3, & \beta^n_{3,2}=\frac{1}{n}\alpha_3+(1-\frac{1}{n})\alpha_2,\\
\beta^n_{2,4}=\frac{1}{n}\alpha_2+(1-\frac{1}{n})\alpha_4, & \beta^n_{4,2}=\frac{1}{n}\alpha_4+(1-\frac{1}{n})\alpha_2,\\
\beta^n_{3,4}=\frac{1}{n}\alpha_3+(1-\frac{1}{n})\alpha_4, & \beta^n_{4,3}=\frac{1}{n}\alpha_4+(1-\frac{1}{n})\alpha_3.
\end{array}$$

则

$$H(\beta^n_{1,2})=\{\beta^n_{1,2},\beta^n_{2,1},\beta^n_{1,3},\beta^n_{3,1},\beta^n_{1,4},\beta^n_{4,1},\beta^n_{2,3},\beta^n_{3,2},\beta^n_{2,4},\beta^n_{4,2},\beta^n_{3,4},\beta^n_{4,3}\},$$

$n=1,2,\cdots,$ 且

$$\Omega=\bigcup_{i=1}^{\infty}H(\beta^n_{1,2}),\ H(\beta^n_{1,2})\cap H(\beta^m_{1,2})=\emptyset,\ n\neq m.$$

齐性空间: 如果群$G$在集合$\Omega$上的作用只有一条轨道, 即对于任意$x,y\in\Omega$, 存在$g\in G$, 使得$y=g\circ x$, 则群$G$在集合$\Omega$上的这个作用是传递的, 此时$\Omega$称为群$G$的齐性空间.

例如, 若$\Omega=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$, 则$H(\alpha_1)=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}=\Omega$, 故$\Omega$是$H$的齐性空间.

定义 设群$G$在集合$\Omega$上有一个作用. 给定$x\in\Omega$, 令$G_x=\{g\in G|g\circ x=x\}$, 则称$G_x$为$x$的稳定子.

易证: $G_x<G$, 称为$x$的稳定子群.

例如, $G_{\alpha_i}=\{I,A_i,A_i^2\}$, $i=1,2,3,4$, $G_{\alpha_j}=G_{\overline{\alpha}_j}=\{I,B_j\},j=5,6,7$.

## 三. Sylow定理的例子

Sylow第一定理 设群$G$的阶为$n = p^l m$, 其中$p$为素数, $(m,p) = 1, l > 0$, 则对于$1 \leq k \leq l$, $G$中必有$p^k$阶子群, 其中$p^l$阶子群称为$G$的Sylow $p-$子群.

对于群$H$, $|H| = 12$, 有4个3阶子群: $H_1 = \{I, A_1, A_1^2\}$, $H_2 = \{I, A_2, A_2^2\}$, $H_3 = \{I, A_3, A_3^2\}$, $H_4 = \{I, A_4, A_4^2\}$; 3个2阶子群: $H_5 = \{I, B_1\}$, $H_6 = \{I, B_2\}$, $H_7 = \{I, B_3\}$; 1个4阶子群: $H_8 = \{I, B_1, B_2, B_3\}$; 没有6阶子群. 2, 3, 4都是素数的方幂.

Sylow 第二定理 设群$G$的阶为$n=p^lm$, 其中$p$为素数, $(m,p)=1$, $l>0$, 则

(1) 对于$1\leq k\leq l$, $G$的任意一个$p^k$阶子群一定包含在$G$的某个Sylow $p-$子群中;

(2) $G$的任意两个Sylow $p-$子群在$G$中共轭.

考虑群$H$, 则$(A_2^2)^TH_1A_2^2=H_4$, $B_1^TH_1B_1=H_3$, $B_3^TH_1B_3=H_2$. 由此可见, $H$的Sylow 3-子群都是共轭的, 且2阶子群都包含在Sylow 2-子群$H_8$中.

Sylow第三定理 设群$G$的阶为$n=p^lm$, 其中$p$为素数, $(m,p)=1$, $l>0$, $G$的Sylow $p-$子群的个数为$r$, 则$r\equiv 1 \pmod p$, 且$r|m$.

例如, $H$中3-Sylow子群的个数为4, 2-sylow子群的个数为1.

谢谢!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